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Research Article

Massive Conformal Gravity

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We construct a massive theory of gravity that is invariant under conformal transformations. The massive action of the theory depends on the metric tensor and a scalar field, which are considered the only field variables. We find the vacuum field equations of the theory and analyze its weak-field approximation and Newtonian limit.

1. Introduction

The study of massive gravity started with Fierz and Pauli [1], who constructed an action describing a free massive spin-2 particle in flat spacetime. It was realized later that the Fierz-Pauli theory coupled to a source is different from linearized general relativity in the massless limit [2, 3]. This is known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity. In order to cure this discontinuity, Vainshtein proposed adding nonlinear effects to the Fierz-Pauli theory [4]. The Vainshtein theory, however, has an extra degree of freedom known as the Boulware-Deser (BD) ghost [5]. Some nonlinear massive gravity theories developed recently [6, 7] eliminate the BD ghost but give rise to unstable cosmological solutions [8]. The solution of this problem leads to massive gravity theories where the Lorentz invariance is broken [9].

It is well know that the theories of elementary particles are invariant under Lorentz transformations. In addition, these theories present local conformal symmetry. Similarly, it is reasonable to expect that the gravity theory be invariant under coordinate transformations and conformal transformations. The usual procedures to obtain a conformally invariant gravity theory are either to adopt the Weyl action [10] or the Einstein-Hilbert action conformally coupled to a scalar field [11]. Several works based on the Weyl action have been carried out in the literature (see, e.g., [12–14]).

In this paper we address a conformally invariant massive gravity theory based on both the Weyl action and the Einstein-Hilbert action conformally coupled to a scalar field. In Section 2 we construct the massive conformal gravitational action and derive the vacuum field equations of the theory. In Section 3 we investigate the limit of the theory in which the fields are weak. In Section 4 we find the Newtonian limit of the theory. Finally, in Section 5 we present our conclusions.

2. Massive Gravity with Conformal Invariance

A conformal transformation is a change of the spacetime geometry that alters the length scales. The conformal transformation of the spacetime metric is defined by

$$\tilde{g}_{\mu\nu} = e^{2\theta(x)} g_{\mu\nu}, \tag{1}$$

where $\theta(x)$ is an arbitrary function of the spacetime coordinates. With the help of (1), it is possible to verify that the Weyl tensor

$$\begin{aligned} C^{\alpha}_{\mu\beta\nu} = {} & R^{\alpha}_{\mu\beta\nu} + \frac{1}{2}\left(\delta^{\alpha}_{\nu} R_{\mu\beta} - \delta^{\alpha}_{\beta} R_{\mu\nu} + g_{\mu\beta} R^{\alpha}_{\nu} - g_{\mu\nu} R^{\alpha}_{\beta}\right) \\ & + \frac{1}{6}\left(\delta^{\alpha}_{\beta} g_{\mu\nu} - \delta^{\alpha}_{\nu} g_{\mu\beta}\right) R \end{aligned} \tag{2}$$

is conformally invariant, where $R^{\alpha}_{\mu\beta\nu}$ is the Riemann tensor, $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ is the Ricci tensor, and $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature.

The square of the Weyl tensor leads to the unique gravitational action constructed out of the metric tensor only that is invariant under conformal transformations. It is given by

$$S_g = -\frac{\alpha}{2kc}\int d^4x\sqrt{-g}\left(C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu}\right), \tag{3}$$

where α is a dimensionless constant and $k = 16\pi G/c^4$ (G is the gravitational constant and c is the speed of light in vacuum). The Weyl action (3) is of fourth order with respect to the metric derivatives. However, by introducing a scalar field it is possible to construct conformally invariant gravitational actions having at most second order derivatives of the metric. The simplest of such actions reads as

$$S_g = \frac{\beta}{2kc}\int d^4x\sqrt{-g}\left(\varphi^2 R + 6\partial_\mu\varphi\partial^\mu\varphi\right), \tag{4}$$

where β is a dimensionless constant and φ is a scalar field that transforms as

$$\tilde{\varphi} = e^{-\theta}\varphi \tag{5}$$

under a conformal transformation. Note that action (4) is invariant under conformal transformations after the appropriate integration of the boundary term [15].

Actions (3) and (4) are the main candidates to form a massive gravitational action with conformal symmetry. In analogy with other massive theories, it is expected that the mass term of a massive gravitational action be of lower order with respect to the metric derivatives than the massless term of the action. Thus a natural choice of a conformally invariant massive gravitational action is given by

$$\begin{aligned} S_g = -\frac{1}{2kc}\int d^4x\sqrt{-g}\,&\left[\alpha C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu}\right. \\ &\left. -\beta\lambda^{-2}\left(\varphi^2 R + 6\partial_\mu\varphi\partial^\mu\varphi\right)\right], \end{aligned} \tag{6}$$

where $\lambda = \hbar/mc$ ($\hbar$ is the Planck constant and m is the graviton mass).

Varying action (6) with respect to $g^{\mu\nu}$ and φ in vacuum, we obtain the field equations

$$\begin{aligned} 2\alpha W_{\mu\nu} - \beta\lambda^{-2}&\left[\varphi^2 G_{\mu\nu} + 6\partial_\mu\varphi\partial_\nu\varphi\right. \\ &\left. -3g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi + g_{\mu\nu}\Box\varphi^2 - \nabla_\mu\nabla_\nu\varphi^2\right] = 0, \end{aligned} \tag{7}$$

$$\Box\varphi - \frac{1}{6}R\varphi = 0, \tag{8}$$

respectively, where

$$W_{\mu\nu} = \nabla^\alpha\nabla^\beta C_{\mu\alpha\nu\beta} - \frac{1}{2}R^{\alpha\beta}C_{\mu\alpha\nu\beta} \tag{9}$$

is the Bach tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{10}$$

is the Einstein tensor, and

$$\Box\varphi = \nabla_\rho\nabla^\rho\varphi = \frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}\partial^\rho\varphi\right) \tag{11}$$

is the generally covariant d'Alembertian for a scalar field. The field equations (7) and (8) are rather intricate, and it is not easy to find any simple solution of these equations. However, the Newtonian limit of the theory yields a simple and interesting solution, as we will see in Section 4.

3. The Weak-Field Approximation

By imposing the weak-field approximations

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{12}$$

$$\varphi = \varphi^{(0)}(1+\sigma) = \sqrt{\frac{2\alpha}{\beta}}(1+\sigma), \tag{13}$$

to (7) and (8), and neglecting terms of second order in $h_{\mu\nu}$ and σ, we obtain the linearized field equations

$$\begin{aligned} &\partial_\rho\partial^\rho\overline{R}_{\mu\nu} - \frac{1}{3}\partial_\mu\partial_\nu\overline{R} - \frac{1}{6}\eta_{\mu\nu}\partial_\rho\partial^\rho\overline{R} \\ &\quad - \lambda^{-2}\left(\overline{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{R} - 2\partial_\mu\partial_\nu\sigma + 2\eta_{\mu\nu}\partial_\rho\partial^\rho\sigma\right) = 0, \end{aligned} \tag{14}$$

$$\partial_\rho\partial^\rho\sigma - \frac{1}{6}\overline{R} = 0, \tag{15}$$

respectively, where

$$\overline{R}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial^\sigma h_{\sigma\nu} + \partial_\nu\partial^\sigma h_{\sigma\mu} - \partial_\sigma\partial^\sigma h_{\mu\nu} - \partial_\mu\partial_\nu h\right) \tag{16}$$

is the linearized Ricci tensor and

$$\overline{R} = \partial^\mu\partial^\nu h_{\mu\nu} - \partial_\rho\partial^\rho h \tag{17}$$

is the linearized scalar curvature, with $h = h^\rho_\rho = \eta^{\mu\nu}h_{\mu\nu}$.

The linearized field equations (14) and (15) are invariant under the coordinate gauge transformation

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \tag{18}$$

where ξ^μ is an arbitrary spacetime dependent vector field, and under the conformal gauge transformations

$$\begin{aligned} h_{\mu\nu} &\longrightarrow h_{\mu\nu} + \eta_{\mu\nu}\Lambda, \\ \sigma &\longrightarrow \sigma - \frac{1}{2}\Lambda, \end{aligned} \tag{19}$$

where Λ is an arbitrary spacetime dependent scalar field.

We may impose the coordinate gauge condition

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0, \tag{20}$$

which fixes the coordinate gauge freedom up to a residual coordinate gauge parameter satisfying $\partial_\rho\partial^\rho\xi_\mu = 0$ and the conformal gauge condition: (We can instead impose the unitary gauge $\sigma = 0$. These two gauge conditions give the same classical results, as we will see in the next section.

However, the unitary gauge is not suitable for a quantum analysis [16], since it breaks the conformal symmetry.)

$$\partial^{\mu}\partial^{\nu}h_{\mu\nu}-\partial_{\rho}\partial^{\rho}h-6\lambda^{-2}\sigma=0, \tag{21}$$

which fixes the conformal gauge freedom up to a residual conformal gauge parameter satisfying $(\partial_{\rho}\partial^{\rho}-\lambda^{-2})\Lambda=0$. Combining (14), (15), (20), and (21), we arrive at

$$\left(\partial_{\rho}\partial^{\rho}-\lambda^{-2}\right)\partial_{\sigma}\partial^{\sigma}h_{\mu\nu}=0, \tag{22}$$

$$\left(\partial_{\rho}\partial^{\rho}-\lambda^{-2}\right)\sigma=0. \tag{23}$$

These two wave equations describe eight degrees of freedom: five for a massive spin-2 particle, two for a massless spin-2 particle, and one for a massive spin-0 particle.

The momentum space propagators of (22) and (23) are given by

$$D_{\mu\nu,\alpha\beta}(k)=\frac{(-i/2)\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}\right)}{k^{2}\left(k^{2}+\lambda^{-2}\right)}, \tag{24}$$

$$D(k)=\frac{-i}{k^{2}+\lambda^{-2}}, \tag{25}$$

respectively. These propagators have a good ultraviolet behavior, so the standard power counting arguments can be used. Note that we can write the propagator (24) as

$$D_{\mu\nu,\alpha\beta}(k)=\frac{(-i/2)\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}\right)}{\lambda^{-2}}\times\left[\frac{1}{k^{2}}-\frac{1}{k^{2}+\lambda^{-2}}\right]. \tag{26}$$

The minus sign between the two terms in brackets suggest the presence of a negative norm ghost state in massive conformal gravity. However, the theory might be free from ghosts if quantized correctly according to the rules of a conformal quantum mechanics. A similar procedure has been carried out with $\mathcal{PT}$ symmetric oscillators by using the methods of $\mathcal{PT}$ quantum mechanics [17]. Thus a careful quantum analysis is necessary on this issue.

4. The Newtonian Limit

The massive conformal gravity must be completely conformal. This means that the general relativistic line element $ds^2=g_{\mu\nu}dx^{\mu}dx^{\nu}$ must be replaced by the conformally invariant line element

$$ds^{2}=\left(\varphi^{2}g_{\mu\nu}\right)dx^{\mu}dx^{\nu}. \tag{27}$$

Accordingly, the interval s between two points P_1 and P_2 along a parametrized timelike curve $x^{\mu}=x^{\mu}(\tau)$ is given by

$$s=\int_{P_1}^{P_2}\left(\varphi^{2}g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\right)^{1/2}d\tau, \tag{28}$$

where the parameter τ is identified as the proper time. The extremization of the functional (28) gives the conformal geodesic equation [18]

$$\frac{d^{2}x^{\lambda}}{d\tau^{2}}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}+\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^{\rho}}\left(g^{\lambda\rho}+\frac{dx^{\lambda}}{d\tau}\frac{dx^{\rho}}{d\tau}\right)=0, \tag{29}$$

where

$$\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}g^{\lambda\rho}\left(\partial_{\mu}g_{\nu\rho}+\partial_{\nu}g_{\mu\rho}-\partial_{\rho}g_{\mu\nu}\right) \tag{30}$$

is the Levi-Civita connection.

The theory presented here is independent of the gauge choice. However, it will be easier to find the classical results of the theory by imposing the unitary gauge $\varphi=\varphi_0=$ constant. In this case, the conformal geodesic equation (29) reduces to

$$\frac{d^{2}x^{\lambda}}{d\tau^{2}}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}=0, \tag{31}$$

which is just the general relativistic geodesic equation. As is well know, the Newtonian limit of such geodesic equation gives

$$h_{00}=-\frac{2\phi}{c^{2}}, \tag{32}$$

where ϕ is the time-independent Newtonian potential.

If we choose $\varphi=\sqrt{2\alpha/\beta}$, for simplicity, the field equations (7) and (8) become

$$W_{\mu\nu}-\lambda^{-2}G_{\mu\nu}=0, \tag{33}$$

$$R=0, \tag{34}$$

respectively. Taking into account the weak-field approximation (12) and the coordinate gauge condition (20), it is not difficult to see that (33) and (34) lead to the wave equation (22). For a time-independent field, the 00 component of this wave equation reduces to

$$\left(\nabla^{2}-\lambda^{-2}\right)\nabla^{2}h_{00}=0, \tag{35}$$

where ∇^2 is the Laplacian operator.

Substituting (32) into (35), we obtain

$$\left(\nabla^{2}-\lambda^{-2}\right)\nabla^{2}\phi=0. \tag{36}$$

The solution of this equation in spherical coordinates reads as

$$\phi(r)=a+\frac{b}{r}+c\frac{e^{-r/\lambda}}{r}+d\frac{e^{r/\lambda}}{r}, \tag{37}$$

where a, b, c, and d are arbitrary constants. Since $\lambda>0$, the last term in (37) goes to infinity as $r\to\infty$, which is unphysical. In addition, at a small distance ($r\ll\lambda$) from a particle of mass M, the potential (37) must reduce to the usual Newtonian potential

$$\phi(r)=-\frac{GM}{r}. \tag{38}$$

So we set $a = 0$, $b = -GM/(1+\gamma)$, $c = -\gamma GM/(1+\gamma)$, and $d = 0$, and (37) becomes

$$\phi(r) = -\frac{GM}{r(1+\gamma)}\left[1+\gamma e^{-r/\lambda}\right], \tag{39}$$

where γ determines the strength of the Yukawa potential relative to the Newtonian potential.

The constant γ and the range λ of the Yukawa potential must be determined by experimental tests. The rotation curves of the major number of galaxies are reproduced with $\gamma = -0.92$ and $\lambda = 20$–$30\,\mathrm{kpc}$, which requires that $m \sim 10^{-26}\,\mathrm{eV/c^2}$ [19, 20]. The maximum length scale of galaxies is in some way determined by λ. On scales larger than λ the repulsive Yukawa potential cuts off and the attractive Newtonian potential remains, which allows the formation of galaxy clusters.

It is worth noting that the gravitational potential present here is not related with the Newtonian solutions discussed in the work of Flanagan [21] and later work of Mannheim [22]. The theory of conformal gravity with dynamical mass generation considered by the authors leads to

$$R = \frac{6\rho}{\varphi_0^2 c} \tag{40}$$

in vacuum, where ρ is the source density. We can readily see that this equation differs from (34) of massive conformal gravity with the unitary gauge $\varphi = \varphi_0$ imposed.

5. Final Remarks

The theory presented here might play an important role on both atomic and cosmological scales. The use of the correct conformal quantization method may show that the theory is renormalizable and unitary. At the same time, the modified potential (39) seems to be a good candidate to describe cosmological phenomena such as the galaxies rotation curves. These issues are under investigation now. The coupling of the theory with matter fields, which is important for a complete description of the theory, will be investigated in the future.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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